Calculation for the 3-DOF Tilt Sensor

The Earth's magnetic field is described by seven parameters. These are declination (D), inclination (I), horizontal intensity (H), vertical intensity (Z), total intensity (F) and the north (X) and east (Y) components of the horizontal intensity. [1]

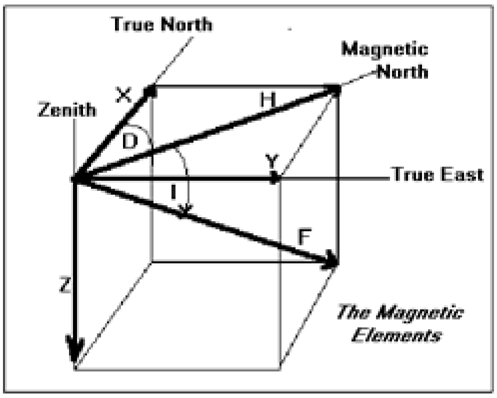

Figure 1 The magnetic elements [1]

Magnetic inclination is the angle that a magnetic needle (i.e. direction of magnetic vector) makes with the plane of the horizon. In Bangkok, the magnetic inclination is approximately 14 degree [2], which means the gravity and the magnetic vectors are not exactly perpendicular. From the directions of magnetic vector and gravity vector, coordinate frame 0 $\left(x_0, y_0, z_0\right)$ is form as shown in Figure 2. The coordinate frame 1 $\left(x_1, y_1, z_1\right)$ is the coordinate frame of the sensor. The measurement of sensors is the components of magnetic vector $\mathbf{m}$ and the gravity vector $\mathbf{g}$ with respect to the coordinate frame 1 (the sensor frame).

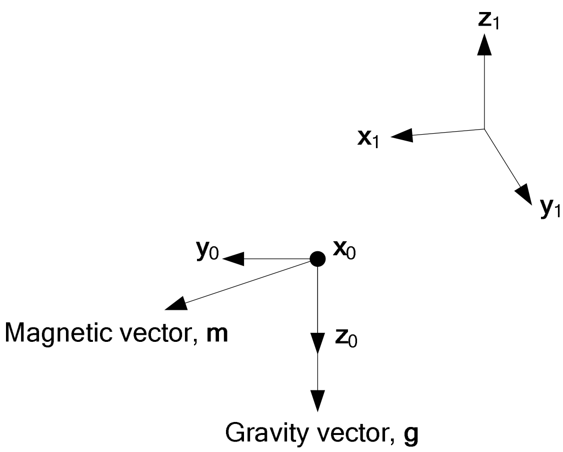

Figure 2 Coordinate definitions

The rotation matrix between coordinate frame 0 and 1 can be constructed by computing the direction of x, y and z axis of frame 0 with respect to frame 1. Since the z axis of frame 0 is in the same direction as the vector $\mathbf{g}$, the unit normal vector in this direction is given by

$$\mathbf{z}_0^1 = \frac{\mathbf{g}}{\|\mathbf{g}\|}$$

The x axis of frame 0 is normal to both vector $\mathbf{m}$ and vector $\mathbf{g}$. The unit normal vector pointing in x axis can be given by

$$\mathbf{x}_0^1 = \frac{\mathbf{m}\times\mathbf{g}}{\|\mathbf{m}\times\mathbf{g}\|}$$

Then, the y axis of frame 0 is automatically defined from x and z axis, which can be obtained by

$$\mathbf{y}_0^1 = \mathbf{z}_0^1 \times \mathbf{x}_0^1$$

The rotation matrix from coordinate frame 1 to coordinate frame 0 can be constructed by

$$\mathbf{R}_0^1 = \begin{bmatrix} \mathbf{x}_0^1 & \mathbf{y}_0^1 & \mathbf{z}_0^1 \end{bmatrix}$$

$$\mathbf{R}_1^0 = \left(\mathbf{R}_0^1\right)^{-1} = \left(\mathbf{R}_0^1\right)^T = \begin{bmatrix} n_x & t_x & b_x \\ n_y & t_y & b_y \\ n_z & t_z & b_z \end{bmatrix}$$

Next, we want to compute the angles of rotation from this rotation matrix. The angles of rotation are defined by using the following consecutive rotations (see Figure 3).

1. Rotation around z axis with the angle θ_1
2. Rotation around the new x’ axis with the angle θ_2
3. Rotation around the new z’’ axis with the angle θ_3

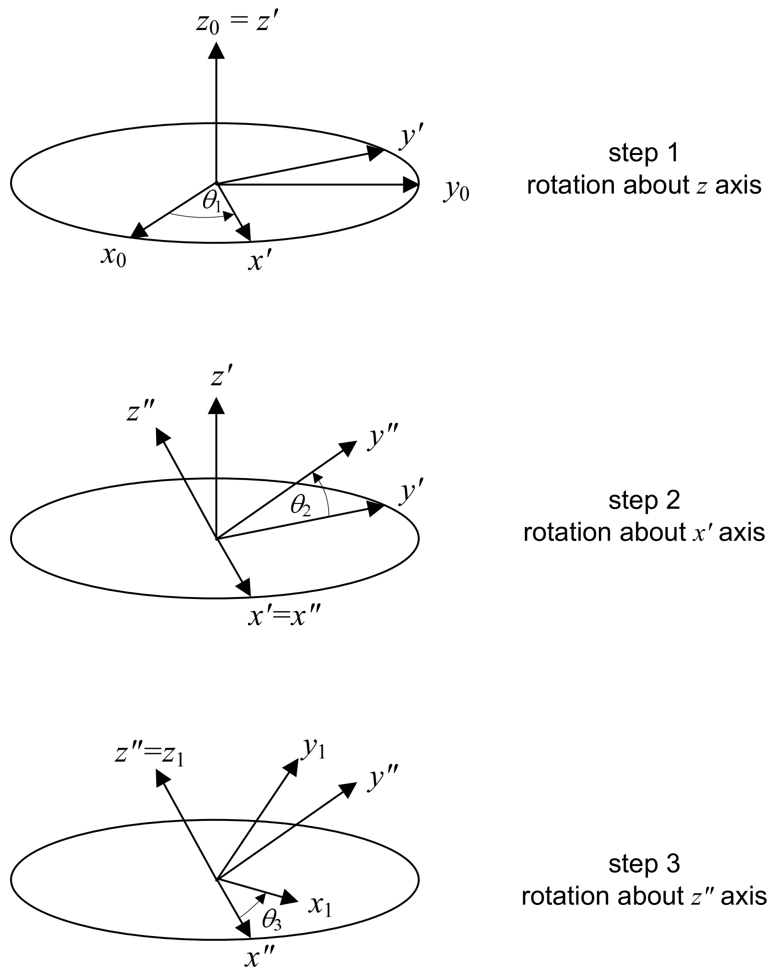

Figure 3 Definition of the angles of rotation

These 3 angles, θ_1, θ_2 and θ_3, uniquely determine the orientation of the coordinate frame 1 with respect to coordinate frame 0. This set of rotation angles are commonly referred to as *Euler angles*. The following are the rotation matrices corresponding to the rotation around z, x and z axis, respectively:

$$\mathbf{R}_z(\theta_1) = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R}_x(\theta_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{bmatrix}, \mathbf{R}_z(\theta_3) = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the consecutive rotation as defined above will result in the combined rotation matrix as follows.

$$\mathbf{R}_z(\theta_1)\mathbf{R}_x(\theta_2) = \begin{bmatrix} c_1 & -s_1c_2 & s_1s_2 \\ s_1 & c_1c_2 & -c_1s_2 \\ 0 & s_2 & c_2 \end{bmatrix}$$

and

$$\mathbf{R}_1^0 = \mathbf{R}_z(\theta_1)\mathbf{R}_x(\theta_2)\mathbf{R}_z(\theta_3) = \begin{bmatrix} * & * & s_1 s_2 \\ * & * & -c_1 s_2 \\ s_2 s_3 & s_2 c_3 & c_2 \end{bmatrix} = \begin{bmatrix} n_x & t_x & b_x \\ n_y & t_y & b_y \\ n_z & t_z & b_z \end{bmatrix} \quad (1)$$

By equating the elements of the matrices in (1), we can compute the angle of rotation as follows.

$\frac{R(1,3)}{R(2,3)}$; $$\frac{s_1 \cancel{s_2}}{-c_1 \cancel{s_2}} = \frac{b_x}{b_y} \rightarrow \theta_1 = \tan^{-1}\left(\frac{b_x}{-b_y}\right)$$

$\frac{R(3,1)}{R(3,2)}$; $$\frac{\cancel{s_2} s_3}{\cancel{s_2} c_3} = \frac{n_z}{t_z} \rightarrow \theta_3 = \tan^{-1}\left(\frac{n_z}{t_z}\right)$$

$s_1 \cdot R(1,3) - c_1 \cdot R(2,3)$; $$s_1^2 s_2 + c_1^2 s_2 = s_1 b_x - c_1 b_y$$

$$s_2 = s_1 b_x - c_1 b_y \quad (2)$$

$R(3,3)$; $$c_2 = b_z \quad (3)$$

(2) ÷ (3); $$\frac{s_2}{c_2} = \frac{s_1 b_x - c_1 b_y}{b_z} \rightarrow \theta_2 = \tan^{-1}\left(\frac{s_1 b_x - c_1 b_y}{b_z}\right)$$

In conclusion, The set of equations used to compute the angles of rotation are

$$\boxed{\theta_1 = \tan^{-1}\left(\frac{b_x}{-b_y}\right),\ \theta_2 = \tan^{-1}\left(\frac{s_1 b_x - c_1 b_y}{b_z}\right) \text{ and } \theta_3 = \tan^{-1}\left(\frac{n_z}{t_z}\right)}$$

References

[1] National Geophysical Data Center (NGDC) Website, "Further Understanding of Geomagnetism," http://www.ngdc.noaa.gov/seg/geomag/geomaginfo.shtml

[2] National Geophysical Data Center (NGDC) Website, "Maps of Magnetic Elements from the WMM 2005," http://www.ngdc.noaa.gov/seg/WMM/image.shtml